

# On the reduction of Feynman integrals to master integrals

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**A.V. Smirnov<sup>\*a</sup> and V.A. Smirnov<sup>b</sup>**

<sup>a</sup> *Scientific Research Computing Center of Moscow State University, Moscow, Russia*

<sup>b</sup> *Nuclear Physics Institute of Moscow State University, Moscow, Russia*

*E-mail: asmirnov@rdm.ru, smirnov@theory.sinp.msu.ru*

The reduction of Feynman integrals to master integrals is an algebraic problem that requires algorithmic approaches at the modern level of calculations. Straightforward applications of the classical Buchberger algorithm to construct Gröbner bases seem to be inefficient. An essential modification designed especially for this problem has been worked out. It has been already applied in two- and three-loop calculations. We are also suggesting to combine our method with the Laporta's algorithm.

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<sup>\*</sup>Speaker.

## 1. Introduction. Reduction problem for Feynman integrals

After a tensor reduction is performed a given Feynman graph  $\Gamma$  results into various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators

$$F(a_1, \dots, a_n) = \int \dots \int \frac{d^d k_1 \dots d^d k_h}{E_1^{a_1} \dots E_n^{a_n}}. \quad (1.1)$$

$d = 4 - 2\varepsilon$ ; the denominators  $E_r$  are either quadratic or linear with respect to the loop momenta  $p_i = k_i$ ,  $i = 1, \dots, h$  and to the independent external momenta  $p_{h+1} = q_1, \dots, p_{h+N} = q_N$  of the graph.

There are different methods of evaluating those integrals. An old analytical strategy is to evaluate, by some methods, every scalar Feynman integral generated by the given graph. And what has already become a traditional strategy is to derive, without calculation, and then apply integration by parts (IBP) relations [7] between the given family of Feynman integrals as recurrence relations.

After such a reduction a general integral from the family (1.1) is expressed as a linear combination of some basic (*master*) integrals. Thus, the whole problem of evaluation falls apart into two parts: constructing a reduction procedure and evaluating master integrals. This paper is devoted to the first part of the problem.

Let us introduce some notation. Let  $\mathcal{F} = F(a_1, \dots, a_n)$  be functions of integer variables  $a_1, \dots, a_n$ ; it is an infinitely dimensional linear space. The simplest basis consists of elements

$$H_{a_1, \dots, a_n}(a'_1, \dots, a'_n) = \delta_{a_1, a'_1} \dots \delta_{a_n, a'_n}. \quad (1.2)$$

From this point of view, Feynman integrals form a point in  $\mathcal{F}$ ; all relations we use turn into linear functionals on this vector space.

There are different types of relations. The most commonly used among them are IBP relations [7]

$$\int \dots \int d^d k_1 d^d k_2 \dots \frac{\partial}{\partial k_i} \left( p_j \frac{1}{E_1^{a_1} \dots E_n^{a_n}} \right) = 0, \quad (1.3)$$

After differentiation we obtain some relations of the following form:

$$\sum \alpha_i F(a_1 + b_{i,1}, \dots, a_n + b_{i,n}) = 0. \quad (1.4)$$

There are more relations one can consider: Lorentz-invariance (LI) identities [8], symmetry relations, e.g.,

$$F(a_1, \dots, a_n) = (-1)^{d_1 a_1 + \dots + d_n a_n} F(a_{\sigma(1)}, \dots, a_{\sigma(n)}), \quad (1.5)$$

boundary conditions: i.e. the conditions of the following form:

$$F(a_1, a_2, \dots, a_n) = 0 \text{ when } a_{i_1} \leq 0, \dots, a_{i_k} \leq 0 \quad (1.6)$$

for some subsets of indices  $i_j$ ; parity conditions and others. All those relations lead to a possibility to express given Feynman integrals in terms of other Feynman integrals.

Therefore one has to name certain integrals irreducible (*master*) and aim to reduce any other integral to those. An attempt to formalize the definition of master integrals was made in [17]. To define master integrals one needs to introduce a priority between the points  $(a_1, \dots, a_n)$ , hence an *ordering*.

There are different ways to choose an ordering. To start with, Feynman integrals are simpler, from the analytic point of view, if they have more non-positive indices. Moreover, solving IBP relations by hand usually consists of reducing indices to zero.

Naturally, we come to the notion of *sectors*. Those are  $2^n$  regions labeled by subsets  $\nu \subseteq \{1, \dots, n\}$ , where  $\sigma_\nu = \{(a_1, \dots, a_n) : a_i > 0 \text{ if } i \in \nu, a_i \leq 0 \text{ if } i \notin \nu\}$ . A sector  $\sigma_\nu$  is said to be *lower* than a sector  $\sigma_\mu$  if  $\nu \subset \mu$ . Furthermore,  $F(a_1, \dots, a_n) \succ F(a'_1, \dots, a'_n)$  if the sector of  $(a'_1, \dots, a'_n)$  is lower than the sector of  $(a_1, \dots, a_n)$ . To define an ordering completely one has to introduce it in some way inside the sectors (this will be discussed below).

Initially they used to solve relations by hand, but with the growth of the complexity of the problems it turned to be impractical. The first algorithmic approach to solving IBP relations was the so-called Laporta's algorithm [9, 10, 13, 14] which is now very well known and actively used in practice. It is based on the fact that the total number of IBP equations written for concrete indices grows faster than the number of independent Feynman integrals, hence this system of equations sooner or later becomes overdetermined, and one obtains a possibility to perform a reduction to master integrals. At the moment, there is one public implementation of this algorithm called AIR [1] and a lot of private versions.<sup>1</sup>

There are other algorithmic approaches to solving the relations such as the Baikov's method [2, 3, 4, 5] and the use of Gröbner bases [6]. The reduction using Gröbner bases has been suggested by Tarasov [20, 21] who reduced the reduction problem to differential equations by introducing a mass for every line,  $a_i \mathbf{i}^+ \rightarrow \frac{\partial}{\partial m_i^2}$ . A direct application of Gröbner bases (without the use of differential equations) was initially suggested by Gerdt [11].

One more approach initially based on Gröbner bases, called the *s*-bases, has been developed in [16] (see also [18] for a brief review). It was applied in practice for a reduction of a family of three-loop Feynman integrals necessary for the analysis of decoupling of *c*-quark loops in *b*-quark HQET [12]. In the next section we outline basic ideas of this approach in an updated form. Then, in Section 3, we formulate how it can be combined with other strategies, in particular with the Laporta's algorithm.

## 2. The *s*-bases approach

Suppose first that we are interested in expressing any integral in the positive sector  $\sigma_{\{1, \dots, n\}}$  as a linear combination of a finite number of integrals in it. The left-hand sides of IBP relations (1.4) can be expressed in terms of operators of multiplication  $A_i$  and shift operators  $Y_i = \mathbf{i}^+, Y_i^- = \mathbf{i}^-$ ,

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<sup>1</sup>by T. Gehrmann and E. Remiddi, M. Czakon, P. Marquard and D. Seidel, Y. Schröder, C. Sturm, A. Onishchenko, O. Veretin, ...

where

$$(A \cdot F)(a_1, \dots, a_n) = a_i F(a_1, \dots, a_n) \text{ and} \quad (2.1)$$

$$(Y_i^\pm \cdot F)(a_1, \dots, a_n) = F(a_1, \dots, a_i \pm 1, \dots, a_n). \quad (2.2)$$

Let  $\mathcal{A}_{1,\dots,n}$  be the algebra generated by shift operators  $Y_i^\pm$  and multiplication operators  $A_i$ . It acts on the field of functions  $\mathcal{F}$  of  $n$  integer variables. One can choose certain elements  $f_i$  corresponding to IBP relations and write

$$(f_i \cdot F)(a_1, \dots, a_n) = 0. \quad (2.3)$$

The choice of elements  $f_i$  is not unique, we will choose them so that they do not depend on  $Y_i^-$ . Consider the left ideal  $\mathcal{J} \subset \mathcal{A}_{1,\dots,n}$ , generated by the elements  $f_i$ . This ideal is named as the ideal of IBP relations. For any element  $X \in \mathcal{J}$  we have

$$(XF)(1, 1, \dots, 1) = 0. \quad (2.4)$$

Also we have

$$F(a_1, a_2, \dots, a_n) = (Y_1^{a_1-1} \dots Y_n^{a_n-1} F)(1, 1, \dots, 1). \quad (2.5)$$

The idea of the algorithm is to reduce the operator on the right-hand side of (2.5) using the elements of the ideal  $\mathcal{J}$ . Suppose we are interested in  $F(a_1, a_2, \dots, a_n)$ . The reduction problem becomes equivalent to reducing the monomial  $Y_1^{a_1-1} \dots Y_n^{a_n-1}$  modulo the ideal of the IBP relations. After obtaining an expression like

$$Y_1^{a_1-1} \dots Y_n^{a_n-1} = \sum r_i f_i + \sum c_{i_1, \dots, i_n} Y_1^{i_1-1} \dots Y_n^{i_n-1} \quad (2.6)$$

it is left to apply (2.6) to  $F$  at  $a_1 = 1, \dots, a_n = 1$  and obtain the following expression:

$$F(a_1, a_2, \dots, a_n) = \sum c_{i_1, \dots, i_n} F(i_1, i_2, \dots, i_n), \quad (2.7)$$

where integrals on the right-hand side are “master integrals” (the formulas (2.3) and (2.5) are used).

To do the reduction we need an ordering of monomials of operators  $Y_i$  or, similarly, an ordering of points  $(a_1, \dots, a_n)$  in the sector: (for two monomials  $M_1 = Y_1^{i_1-1} \dots Y_n^{i_n-1}$  and  $M_2 = Y_1^{j_1-1} \dots Y_n^{j_n-1}$  we have  $(M_1 \cdot F)(1, \dots, 1) \succ (M_2 \cdot F)(1, \dots, 1)$  if and only if  $M_1 \succ M_2$ ). Then the reduction procedure becomes similar to the division of polynomials. But one needs to introduce a proper ordering.

A monomial is defined by its degree, i.e. a set of  $n$  non-negative integers  $(\mathbb{N}^n)$ . Thus defining an ordering on monomials is equivalent to defining an ordering on  $\mathbb{N}^n$ . We require the following properties:

- i) for any  $a \in \mathbb{N}^n$  not equal to  $(0, \dots, 0)$  one has  $(0, \dots, 0) \prec a$ ;
- ii) for any  $a, b, c \in \mathbb{N}^n$  one has  $a \prec b$  if and only if  $a + c \prec b + c$ .

For example, a lexicographical ordering is defined the following way: a set  $(i_1, \dots, i_n)$  is said to be higher than a set  $(j_1, \dots, j_n)$  (denoted by  $(i_1, \dots, i_n) \succ (j_1, \dots, j_n)$ ) if there is  $l \leq n$  such that  $i_1 = j_1, i_2 = j_2, \dots, i_{l-1} = j_{l-1}$  and  $i_l > j_l$ .

Another example is a degree-lexicographical ordering:  $(i_1, \dots, i_n) \succ (j_1, \dots, j_n)$  if  $\sum i_k > \sum j_k$ , or  $\sum i_k = \sum j_k$  and  $(i_1, \dots, i_n) \succ (j_1, \dots, j_n)$  in the sense of the lexicographical ordering.

An ordering can be defined by a non-degenerate  $n \times n$  matrix  $(a_{k,l})$ : for two sets of numbers  $(i_1, \dots, i_n)$  and  $(j_1, \dots, j_n)$  one first compares  $\sum_l i_l a_{1,l}$  and  $\sum_l j_l a_{1,l}$ . If the first number is greater, then the first degree is greater; if the first number is smaller, then the first degree is smaller; and if those numbers are equal we compare  $\sum_l i_l a_{2,l}$  and  $\sum_l j_l a_{2,l}$  and so on.

The following matrices correspond to a lexicographical, a degree-lexicographical and a reverse degree-lexicographical ordering for  $n = 5$ :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Such an approach encounters the following problem: the reduction does not always lead to a reasonable number of irreducible integrals, so one has to build a special basis of the ideal first. Obviously having elements with lowest possible degrees is equivalent to obtaining master integrals with minimal possible degrees. Therefore one needs to build special bases. This can be done by an algorithm based on the Buchberger algorithm with the use of  $S$ -polynomials and reductions [6].

Moreover, one must keep in mind that we are interested in integrals not only in the positive sector. Our algorithm [16] aims to build a set of special bases of the ideal of IBP relations ( $s$ -bases). The basic idea is to consider the algebra  $\mathcal{A}_v$  generated by operators  $A_i$  and operators  $Y_i^+$  for  $i \in v$  and  $Y_i^-$  for  $i \notin v$ . Then for  $\sigma_v$  we again consider the ideal of IBP relations in  $\mathcal{A}_v$ . Now one has to construct *sector bases* ( $s$ -bases), rather than Gröbner bases for all sectors, where an  $s$ -basis for a sector  $\sigma_v$  is a set of elements of  $\mathcal{I}$  which provides the possibility of a reduction to master integrals in  $\sigma_v$  and integrals whose indices lie in lower sectors, i.e.  $\sigma_{v'}$  for  $v' \subset v$ .

This leads to considering many sectors — seemingly the problem becomes harder. But the important simplification is that one is not trying to solve the reduction problem in each sector separately but allows to reduce the integrals in a given sector to lower sectors — similarly to the “by hand” solutions. It is also worth noting that it is most complicated to construct  $s$ -bases for minimal sectors.

The construction of  $s$ -bases is close to the Buchberger algorithm but it can be terminated when the “current” basis already provides us the needed reduction. The basic operations are the same, i.e. calculating  $S$ -polynomials and reducing them modulo current basis, with a chosen ordering.

After constructing  $s$ -bases for all non-trivial sectors one obtains a recursive (with respect to the sectors) procedure to evaluate  $F(a_1, \dots, a_n)$  at any point and thereby reduce a given integral to master integrals. A description of the old version of the algorithm (implemented in `Mathematica`) can be found in [15].

### 3. Combining $s$ -bases with other strategies

Constructing a basis requires a certain skill to find a proper ordering. At a high level (starting from 10 to 12 indices) it turns to be a certain problem. In some sectors the ordering can be found

automatically, in some sectors it is easily done manually, but in some sectors the  $s$ -bases can be hardly constructed. There are several ideas that might allow the algorithm to construct the bases better and we hope that they will be implemented in the future. Currently this is not a problem of cpu time or memory, but an algorithmical one: with a fixed ordering the algorithm either produces a basis quickly, or does not do it at all.

Recently we have worked out one more way to improve the algorithm. Experience shows that  $s$ -bases are constructed easily if the number of non-positive indices in a given sector is small. And what can help is the fact that if the number of non-positive indices is large, there is usually a possibility to perform integration over some loop momentum explicitly in terms of gamma functions for general  $d$ . However, to derive the corresponding explicit formulae, with multiple finite summations, for all necessary cases turns out to be impractical.

In this situation, there is another alternative. Let us distinguish the propagators (and irreducible numerators) involved in such an explicit integration formula. They are associated with a subgraph of the given graph. Let us solve IBP relations for the corresponding subintegral in order to express any such subintegral in term of some master integrals.

Then, after using this reduction procedure, it will be sufficient to use explicit integration formulae only for some boundary values of the indices. This replacement is very simple, without multiple summations. Coefficients in these reduction formulae depend not only on  $d$  and given external parameters but also on the propagators of the given graph, which are external for the subgraph.

It turned out possible to implement the solution of the recursive problem for the subgraph in terms of the Feynman integrals for the whole graph. In this reduction, pure powers of the parameters which are external for the subgraph transform naturally into the corresponding shift operators and their inverse. Integrals which are obtained from initial integrals by an explicit integration over a loop momentum in terms of gamma function usually involve a propagator with an analytic regularization by an amount proportional to  $\varepsilon$ . Our code is applicable also in such situations.

Even if we have not constructed the  $s$ -bases for all sectors we can still run the reduction procedure. However, in some sectors there will be an infinite number of irreducible integrals left.

One of the ideas is to use the Laporta's algorithm in those sectors. Still it leads to several problems in linking two algorithms together. One of the disadvantages is that the reduction works much slower if the number of irreducible integrals goes high.

Another idea is to combine two algorithms inside one code. The  $s$ -bases might give the reductions in all sectors, where a basis can be constructed. And for the remaining sectors the systems of initial IBP's are used like in the Laporta's algorithm.

Let us summarize the ideas of this paper. The  $s$ -bases approach seems to be a good addition to the methods of automatic integral reduction. The bases construction part of the algorithm should still be improved, but even now it is capable of constructing bases for quite complicated cases. A constructed basis in a sector means that in this sector one needs no more real solving of equations — the system of linear equations becomes upper-triangle.

Moreover, it looks like the future is in combining the  $s$ -bases and the Laporta's algorithm. An algorithm named FIRE (Feynman Integral REDuction) combining all mentioned ideas is currently being developed. We plan on performing much more complicated three-loop calculations with the

use of FIRE (together with M. Steinhauser; the project is in progress). Afterwards we intend to make the algorithm public.

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